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C. 1

1. Introduction

The solitary wave is one of the interesting two-dimensional waves that can occur in a liquid contained in a horizontal channel with constant rectangular cross section. It is a stable progressing wave of permanent type characterized by a single symmetrical intumescence whose elevation decreases to zero as the distance from its crest increases to infinity. The amplitude of this wave is not necessarily small compared with the depth of the channel. It was named by Scott Russell [1], [2] who was led to present the first scientific reports (1837, 1844) about such a wave after he observed it in the Edinburgh and Glasgow canal. Since then, the wave has been studied in numerous papers. A brief account of most of the investigations and a short history of the subject constitute part of a paper by Peters and Stoker [3]. Modern systematic versions of the theory based on the stretching technique invented by Friedrichs [4] can be found in papers by Keller [5], Friedrichs and Hyers [6], Littman [7], Peters and Stoker [3] and in Stoker's book [8].

Heretofore the solitary wave has always been regarded as a two-dimensional phenomenon. It is our purpose here to consider three-dimensional solitary waves moving in a horizontal channel that has an arbitrary cross section.

In Section 2 we use a variant of Friedrichs' method to secure an appropriate system of hydrodynamical equations. This system is used in Section 3 to study irrotational solitary

waves in a long channel with arbitrary cross section. A potential function is used in the analysis of the irrotational wave and consequently the method of Section 3 is useless for the investigation of rotational waves. In Section 4 we return to our basic equations and show that they can be used to give an account of solitary waves on a homogeneous running stream, with arbitrary vorticity, in a horizontal channel with arbitrary cross section.

In both the irrotational and rotational cases we find that a first order approximation to the wave profile is given by the classical formula $q \operatorname{sech}^2 \frac{x\ell}{2d}$, as it is for the case of a channel with rectangular section containing a liquid of constant density, without vorticity. In our cases, however, as may be imagined, the relationships between the speed of the wave, the amplitude q, and the scale factor ℓ, are complicated expressions involving the solution ψ of a boundary value problem that is of elliptic type. These relations are best left in the body of the text, where they can be found with other formulas for the components of velocity and pressure. Three-dimensional aspects appear in the dependence of various parameters on the function ψ. The first order approximation to the wave surface exhibits no lateral modification of amplitude; but this modification does appear in the higher order approximations. The second order approximation already presents a wave surface whose amplitude is not constant in a direction perpendicular to the axis of the channel. Thus the

theory presented here introduces a three-dimensional solitary wave — a single intumescence with amplitude modulations in the direction of the long axis of the channel and also in the horizontal perpendicular direction.

2. Formulation

Let a gravitating, incompressible, and inviscid liquid, with constant density δ, be confined to a horizontal straight channel of infinite length and constant cross section. Suppose that the equilibrium free surface of the liquid is planar and that it is parallel to the horizontal x_1x_3-plane of a cartesian reference frame whose x_1-axis is taken parallel to the rigid wall which forms the channel. Let the wall be defined by the equation

$$x_2 = W(x_3)$$

and the free surface given by

$$x_2 = f(x_1, x_3, t) \ .$$

The positive direction of the x_2-axis is supposed to be upwards. Let g denote the gravitational acceleration, let p denote the pressure, and let us use v_1, v_2, v_3 to denote the velocity components of a liquid particle. In terms of these quantities, the elementary theory of hydrodynamics predicts that the motion of the liquid is determined by the continuity equation

(2.1) $$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \;;$$

the momentum equations

(2.2) $$\begin{cases} \dfrac{\partial v_1}{\partial t} + v_1 \dfrac{\partial v_1}{\partial x_1} + v_2 \dfrac{\partial v_1}{\partial x_2} + v_3 \dfrac{\partial v_1}{\partial x_3} = - \dfrac{1}{\delta} \dfrac{\partial p}{\partial x_1} \\ \dfrac{\partial v_2}{\partial t} + v_1 \dfrac{\partial v_2}{\partial x_1} + v_2 \dfrac{\partial v_2}{\partial x_2} + v_3 \dfrac{\partial v_2}{\partial x_3} = - g - \dfrac{1}{\delta} \dfrac{\partial p}{\partial x_2} \;; \\ \dfrac{\partial v_3}{\partial t} + v_1 \dfrac{\partial v_3}{\partial x_1} + v_2 \dfrac{\partial v_3}{\partial x_2} + v_3 \dfrac{\partial v_3}{\partial x_3} = - \dfrac{1}{\delta} \dfrac{\partial p}{\partial x_3} \end{cases}$$

the kinematic boundary conditions

(2.3) $$v_2 = v_3 \frac{\partial W}{\partial x_3} \;;$$

(2.4) $$v_2 = v_1 \frac{\partial f}{\partial x_1} + v_3 \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial t} \;;$$

plus dynamical conditions at the free surface; initial conditions at $t = 0$; and conditions at infinity. The vorticity components are given by

$$\zeta_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \quad \zeta_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \quad \zeta_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \;.$$

The above equations can be written in dimensionless form if we introduce a typical length in the vertical direction, say h; a dimensionless stretching factor $\sqrt{\varepsilon}$; and the dimensionless variables

$$x = \sqrt{\varepsilon}\, x_1 h^{-1}\ , \quad y = x_2 h^{-1}\ , \quad z = x_3 h^{-1}\ , \quad \tau = \sqrt{\varepsilon}\,\sqrt{gh}\ t h^{-1}$$

$$u = v_1 (gh)^{-\frac{1}{2}}, \quad v = \sqrt{\varepsilon}\ v_2 (gh)^{-\frac{1}{2}}, \quad w = \sqrt{\varepsilon}\ v_3 (gh)^{-\frac{1}{2}}, \quad \pi = p(\delta gh)^{-1}$$

$$\eta = f h^{-1}\ , \qquad \omega = W h^{-1}$$

$$\alpha = \zeta_1 \left(\frac{g}{h}\right)^{-\frac{1}{2}}\ , \qquad \beta = \zeta_2 \left(\frac{g}{h}\right)^{-\frac{1}{2}}\ , \qquad \gamma = \zeta_3 \left(\frac{g}{h}\right)^{-\frac{1}{2}}\ .$$

In terms of these variables the basic equations are

(2.5)
$$\varepsilon\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(2.6)
$$\begin{cases} \varepsilon\left[\dfrac{\partial u}{\partial \tau} + u\ \dfrac{\partial u}{\partial x} + \dfrac{\partial \pi}{\partial x}\right] + v\ \dfrac{\partial u}{\partial y} + w\ \dfrac{\partial u}{\partial t} = 0 \\[2ex] \varepsilon\left[\dfrac{\partial v}{\partial \tau} + u\ \dfrac{\partial v}{\partial x} + 1 + \dfrac{\partial \pi}{\partial y}\right] + v\ \dfrac{\partial v}{\partial y} + w\ \dfrac{\partial v}{\partial z} = 0 \\[2ex] \varepsilon\left[\dfrac{\partial w}{\partial \tau} + u\ \dfrac{\partial w}{\partial x} + \dfrac{\partial \pi}{\partial z}\right] + v\ \dfrac{\partial w}{\partial y} + w\ \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

(2.7)
$$v = w\ \frac{\partial \omega}{\partial z}$$

(2.8)
$$\varepsilon\left[\frac{\partial \eta}{\partial \tau} + u\ \frac{\partial \eta}{\partial x}\right] = v - w\ \frac{\partial \eta}{\partial z}$$

$$
(2.9) \qquad \begin{cases} \alpha = \dfrac{1}{\sqrt{\varepsilon}} \left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) \\[2ex] \beta = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\[2ex] \gamma = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{cases} .
$$

The last equations differ from those introduced by Keller [5] as an extension of the stretching procedure utilized by Friedrichs [4] to obtain the two-dimensional shallow water theory by a formal perturbation procedure. Keller applied the stretching process to the horizontal space variables, the time, and the vertical velocity component. The equations above come from applying the stretching process to <u>one</u> space variable, the time, and the <u>two</u> velocity components perpendicular to the direction of the channel. It should be remarked however, that the above equations are in agreement with Keller's equations when $w = 0$ and the remaining dependent variables are independent of z.

Friedrichs and Hyers [6], Littman [7], and Lavrientiev [9] have proved that the equations presented above possess two-dimensional solitary and cnoidal wave solutions which do not depend on the z-variable; and they have shown that it is asymptotically valid to assume for these wave solutions that each dependent quantity can be developed in integral powers of the parameter ε. For example, one such case is:

$$h^{-1}f(x_1,t) = h^{-1}f\left(\frac{hx}{\sqrt{\varepsilon}}, \frac{h\tau}{\sqrt{\varepsilon gh}}\right)$$

$$= \eta(x,\tau,\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k \eta_k(x,\tau) \ .$$

In the sequel we will be interested in the consequences of the same assumption when it is applied to our three-dimensional equations for motion in a channel which is not required to have a rectangular cross section.

3. Irrotational Motion. The Solitary Wave in a Channel with Arbitrary Cross Section

Suppose that the liquid in the channel is at rest until some moment when a disturbance generates a wave which ultimately becomes a steady wave progressing with constant velocity c in the direction of the negative x-axis. For this steady wave it is convenient to use a coordinate system such that the y-axis passes through a crest of fixed identity; and such that the mean departure of the free surface from the xz-plane is zero at infinity. With respect to this moving coordinate system, the wave form does not change with time; and since the liquid possesses zero vorticity when the motion is generated, the motion is forever irrotational. Hence there exists a velocity potential $\phi(x,y,z)$ such that for a liquid particle with coordinates (x,y,z) the velocity components are

$$(3.1) \qquad u = \phi_x \,, \quad v = \phi_y \,, \quad w = \phi_z$$

and ϕ must satisfy

$$(3.2) \qquad \varepsilon\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \,.$$

When u, v, w in (2.6) are replaced by (3.1) the momentum equations yield the energy integral

$$(3.3) \qquad \left| \varepsilon\left[\frac{\phi_x^2}{2} + y + \pi\right] + \frac{1}{2}\,(\phi_y^2 + \phi_z^2) \right|_{P_o}^{P} = 0$$

where P is an arbitrary point in the liquid and P_o is some convenient fixed point. Since we are interested here in a wave of solitary type, let us take P_o to be the point $(-\infty, 0, 0)$ and let us assume that

$$\phi_y(-\infty,0,0) = \phi_z(-\infty,0,0) = 0 \,, \qquad \phi_x(-\infty,0,0) = \frac{c}{\sqrt{gh}}$$

$$\eta(-\infty,z) = 0 \,, \qquad \pi(-\infty,0,0) = 0 \,.$$

Under these assumptions (3.3) is replaced by

$$(3.4) \qquad \left| \varepsilon\left[\frac{\phi_x^2}{2} + y + \pi\right] + \frac{1}{2}\,(\phi_y^2 + \phi_z^2) \right|_{P} = \frac{\varepsilon c^2}{2gh} \,,$$

an equation which can be used for the computation of pressure once ϕ is known. If the pressure is taken to be zero at the

free surface $y = \eta(x,z)$ and P is a point in this surface then (3.4) becomes

$$\varepsilon\left[\eta + \frac{\phi_x^2}{2}\right] + \frac{1}{2}(\phi_y^2 + \phi_z^2) = \frac{\varepsilon c^2}{2gh} , \tag{3.5}$$

a condition to be satisfied at $y = \eta$. In addition to this condition, (2.8) and (3.1) show that

$$\varepsilon\phi_x\eta_x = \phi_y - \phi_z\eta_z \tag{3.6}$$

must also be satisfied at $y = \eta$. Equation (2.7) becomes

$$\phi_y - \phi_z\omega_z = 0 \tag{3.7}$$

which asserts that the normal velocity of the liquid is zero at the rigid channel wall $y = \omega(z)$.

Let us assume next that $\phi(x,y,z)$ and $\eta(x,z)$ can be expanded so that

$$\phi(x,y,z) = \phi_o + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \dots \tag{3.8}$$

and

$$\eta(x,z) = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \dots \quad . \tag{3.9}$$

The idea now is to substitute these expansions in (3.2), (3.5), (3.6), (3.7); and then equate coefficients of like powers of ε. This leads to a sequence of partial differential systems for

the determination of $\phi_o, \phi_1, \phi_2, \ldots$ and $\eta_1, \eta_2, \ldots$. We need to observe here that the quantity c^2/gh in (3.5) must also be replaced by an expansion in powers of ε. Since

$$\frac{c}{\sqrt{gh}} = \phi_x \Bigg|_{\substack{x=-\infty \\ y=0 \\ z=0}} = \Bigg| \phi_{ox} + \varepsilon\phi_{1x} + \varepsilon^2\phi_{2x} + \ldots \Bigg|_{\substack{x=-\infty \\ y=0 \\ z=0}}$$

$$\frac{c}{\sqrt{gh}} = \frac{1}{\sqrt{gh}} [c_o + \varepsilon c_1 + \varepsilon^2 c_2 + \ldots]$$

we have

$$c^2 = c_o + 2\varepsilon c_o c_1 + \varepsilon^2(c_1^2 + 2c_o c_2) + \ldots \quad . \tag{3.10}$$

The zeroth order system comes from substituting $\varepsilon = 0$ in (3.2), (3.5), (3.6) and (3.7). We find that $\phi_o(x,y,z)$ must satisfy

$$\phi_{oyy} + \phi_{ozz} = 0$$

in the domain D (see Fig. 1) which defines the cross section of the channel. This domain is bounded by L: $y = \omega(z)$; and a segment S of the z-axis determined by the points where L intersects $y = 0$. We assume that L is a piecewise smooth curve which crosses $y = 0$ in just two points p_1 and p_2 such that $b = p_2 - p_1 > 0$. The equations (3.5) and (3.6) show that

$$\phi_{oy}^2(x,0,z) + \phi_{oz}^2(x,0,z) = 0$$

$$\phi_{oy}(x,0,z) = 0$$

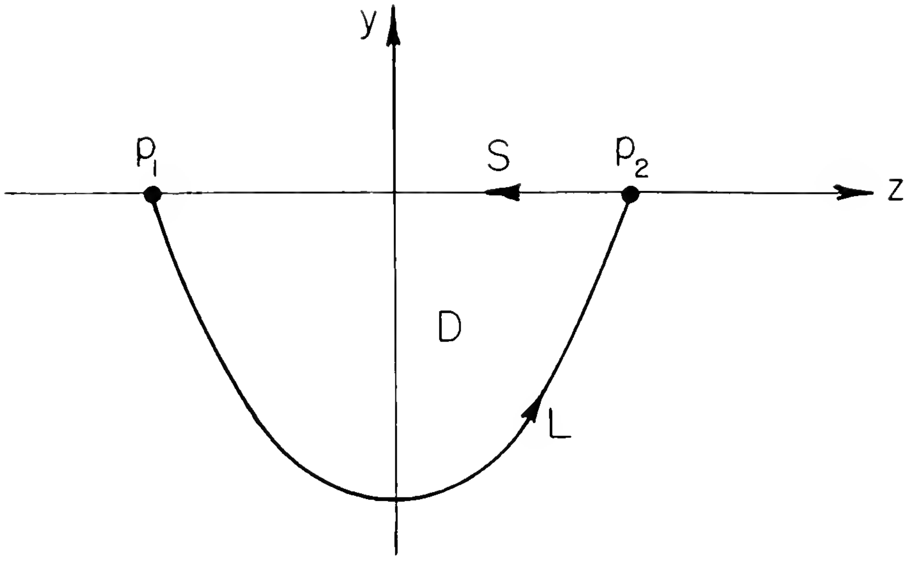

Fig. 1

and (3.7) shows that the normal derivative ϕ_{on} must be zero along L. Since the above requires the normal derivative of the potential function ϕ_o to be zero both along L and S we conclude that

$$\phi_o = F_o(x)$$

is the only solution without singularities at the boundary. In order to fix $F_o(x)$ we must pass to the system of relations of next higher order.

The first order system comes from equating coefficients of ε in the equations (3.2), (3.5), (3.6) and (3.7). The result of doing this shows that ϕ_1 must satisfy

$$\phi_{1yy} + \phi_{1zz} = -F_o''(x)$$

in D. Along S we obtain the conditions

$$\phi_{1y}(x,0,z) = \eta_1(x,z)\phi_{oz}(x,0,z)$$

$$(3.11) \qquad \phi_{ox}^2(x,0,z) + 2\left\{\begin{array}{l} \phi_{oy}(x,0,z)\phi_{1y}(x,0,z) \\ + \phi_{oz}(x,0,z)\phi_{1z}(x,0,z) \end{array}\right\} = \frac{c_o^2}{gh} .$$

In addition, we find that the normal derivative ϕ_{1n} must be zero along L. However, since we know that $\phi_o = F_o(x)$, equation (3.11) shows that

$$F_o'(x) = c_o/\sqrt{gh} ,$$

which means that we can take

$$\phi_o = \frac{c_o x}{\sqrt{gh}} .$$

We see, therefore, that ϕ_1 must be a potential function in D and its normal derivative must be zero on both L and S. It follows again that ϕ_1 is a function of x alone and, since $\phi_{1x}(-\infty,0,0) = c_1/\sqrt{gh}$, we can take

$$\phi_1 = F_1(x) + \frac{c_1 x}{\sqrt{gh}}$$

where $F_1'(-\infty) = 0$.

The second order system shows that ϕ_2 must satisfy

$$\phi_{2yy} + \phi_{2zz} = -F_1''(x)$$

in D and ϕ_{2n} must vanish on L. Along S, ϕ and η must satisfy

$$\phi_{2y}(x,0,z) = \frac{c_o \eta_{1x}}{\sqrt{gh}}$$

(3.12) $$\eta_1 + \frac{c_o}{\sqrt{gh}} F_1'(x) = 0 \ .$$

Thus the normal derivative of ϕ_2 along S is given by

$$\phi_{2y}(x,0,z) = - \frac{c_o^2}{gh} F_1''(x) \ .$$

The constant c_o can be fixed by observing that if χ_n is the outward normal derivative of χ at the boundary of D then

(3.13 $$\iint_D (\chi_{zz} + \chi_{yy}) dxdy = \int_{L+S} \chi_n \, ds$$

where s denotes the arc length along the boundary and L+S is oriented counterclockwise as shoown in Fig. 1. This identity imposes a condition for the existence of a function χ which has a prescribed normal derivative on the boundary of D and which satisfies

$$\chi_{zz} + \chi_{yy} = K(z,y)$$

in D. Such a function does not exist if the assigned normal derivative violates

$$\iint_D K(z,y) dzdy = \int_{L+S} \chi_n \, ds \ .$$

The application of (3.13) to ϕ_2 gives

$$AF_1''(x) = \frac{c_o^2 b}{gh} F_1''(x) \quad .$$

or

$$F_1''(x)\left[A - \frac{c_o^2 b}{gh}\right] = 0 \tag{3.14}$$

where Ah^2 is the area of the cross section of the channel occupied by the liquid at infinity, and bh is the breadth of the free surface at infinity. Equation (3.14) shows that either $F_1''(x) = 0$ or

$$A - \frac{c_o^2 b}{gh} = 0 \ . \tag{3.15}$$

This indicates a bifurcation phenomenon. The first alternative, i.e. $F_1''(x) = 0$, leads to a uniform flow and hence it will be ignored here. The second alternative gives

$$c_o^2 = g\,\frac{Ah}{b} = g\,\frac{Ah^2}{bh} = gd \tag{3.16}$$

where d is the mean depth of the liquid at infinity. Notice that if the vertical scale factor h is taken equal to the mean depth then

$$c_o^2 = gd \quad \text{and} \quad \frac{A}{b} = 1 \ .$$

This choice for h simplifies the equation for ϕ_2. Therefore we will hereafter suppose that

$$h = d \ .$$

In accordance with the terminology used for the case of a channel with rectangular cross section, the speed $c_o = \sqrt{gd}$ will be referred to as the critical speed.

The function ϕ_2 must now satisfy

$$\phi_{2yy} + \phi_{2zz} = -F_1''(x)$$

for (z,y) in D;

$$\phi_{2y}(x,0,z) = -F_1''(x) \tag{3.17}$$

and

$$\phi_{2n} = 0$$

along L. This implies that we can write

$$\phi_2 = -F_1''(x)\psi_2(y,z) + F_2(x) + \frac{c_2 x}{\sqrt{gd}} \tag{3.18}$$

where

$$\psi_{2zz} + \psi_{2yy} = 1$$

for (z,y) in D;

$$\psi_{2y}(0,z) = 1$$

and

$$\psi_{2n} = 0$$

along L. There is no loss in generality if we make ψ_2 unique by specifying $\psi_2(0,0) = 0$. We assume that ψ_2 is now known.

From equation (3.18) we see that

(3.19) $$\eta_1(x,z) = \eta_1(x) = -F_1'(x)$$

but to determine $\eta_1(x)$ we must proceed to the system of third order.

The third order system shows that ϕ_3 must satisfy

(3.20) $$\phi_{3yy} + \phi_{3zz} = F_1'''(x)\psi_2(y,z) - F_2''(x)$$

in D, and ϕ_{3n} must vanish on L. Along S, the requirements are

(3.21) $$\eta_1(x)\phi_{2yy}(x,0,z) + \phi_{3y}(x,0,z)$$

$$= \eta_{2x}(x,z) + \eta_{1x}(x)\left[F_1'(x) + \frac{c_1}{\sqrt{gd}}\right]$$

and

(3.22) $$\eta_2(x,z) + \phi_{2x}(x,0,z) + \frac{1}{2}\left[F_1'(x) + \frac{c_1}{\sqrt{gd}}\right]^2 = \frac{c_1^2}{2gd} + \frac{c_2}{\sqrt{gd}} .$$

In terms of ψ_2 and η_1 these conditions are

(3.23) $$\eta_1(x)\eta_1'(x)\psi_{2yy}(0,z) + \phi_{3y}(x,0,z)$$

$$= \eta_{2x}(x.z) + \eta_1'(x)\left[-\eta_1(x) + \frac{c_1}{\sqrt{gd}}\right]$$

(3.24) $$\eta_2(x,z) + \eta_1''(x)\psi_2(0,z) + F_2'(x) + \frac{\eta_1^2(x)}{2} - \frac{c_1\eta_1(x)}{\sqrt{gd}} = 0 .$$

The elimination of η_2 from (3.21) and (3.22) gives

$$(3.25)\quad \phi_{3y}(x,0,z) = -\eta_1'''(x)\psi_2(0,z) - \eta_1(x)\eta_1'(x)[2+\psi_{2yy}(0,z)] + \frac{2c_1\eta_1'(x)}{\sqrt{gd}} - F_2''(x) .$$

Since the normal derivative ϕ_{3n} is zero on L, the condition (3.13), for $\chi = \phi_3$, can be written

$$\iint_D (\phi_{3zz} + \phi_{3yy})dzdy = \int_{p_1}^{p_2} \phi_{3y}(x,0,z)dz .$$

As we can see from (3.18) and (3.23) this condition requires

$$F_1''''(x) \iint_D \psi_2(y,z)dzdy - AF_2''(x)$$
$$= -\eta_1'''(x)\int_{p_1}^{p_2} \psi_2(0,z)dz - 2b\eta_1(x)\eta_1'(x)$$
$$-\eta_1(x)\eta_1'(x)\int_{p_1}^{p_2} \psi_{2yy}(0,z)dz + \frac{2bc_1\eta_1'(x)}{\sqrt{gd}} - bF_2''(x) .$$

Because $A = b$ and $F_1'(x) = -\eta_1(x)$, the last equation reduces to

$$(3.26)\quad \left[\int_{p_1}^{p_2} \psi_2(0,z)dz - \iint_D \psi_2(z,y)dzdy\right]\eta_1'''(x)$$
$$= \frac{2bc_1\eta_1'(x)}{\sqrt{gd}} - \left[\int_{p_1}^{p_2} \psi_{2yy}(0,z)dz + 2b\right]\eta_1(x)\eta_1'(x)$$

which is an ordinary non linear equation for the determination of the first approximation to the surface wave $\eta(x,z)$ since ψ_2 is known. The coefficient of $\eta_1'''(x)$ can be transformed into

$$m_o = \int_{p_1}^{p_2} \psi_2(0,z)dz - \iint_D \psi_2(y,z)dzdy$$

$$= \int_{L+S} \psi_2\psi_{2n}ds - \iint_D \psi_2[\psi_{2yy} + \psi_{2zz}]dzdy$$

$$= \iint_D (\psi_{2y}^2 + \psi_{2z}^2)dzdy$$

which shows that m_o is positive. The coefficient of $\eta_1(x)\eta_1'(x)$ is

$$m_1 = -\left[\int_{p_1}^{p_2} \psi_{2yy}(0,z)dz + 2b\right]$$

$$m_1 = -\left\{\int_{p_1}^{p_2} [1 - \psi_{2zz}(0,z)]dz + 2b\right\}$$

$$= -3b\left\{1 - \frac{[\psi_{2z}(0,p_2) - \psi_{2z}(0,p_1)]}{3b}\right\}$$

$$= -3br$$

where

$$r = 1 - \frac{[\psi_{2z}(0,p_2) - \psi_{2z}(0,p_1)]}{3b} .$$

We remark that if the curve L intersects the z-axis orthogonally then $r = 1$. [If $\psi_{2z}(0,p_2) - \psi_{2z}(0,p_1)$ is positive one can make $r > 0$ by taking $1/b = d/(\text{breadth})$ sufficiently small.] If in addition to the above definitions of m_o and m_1 we define m_2 as

$$m_2 = \frac{2bc_1}{\sqrt{gd}}$$

the equation (3.26) can be written as

$$(3.27 \qquad m_o\eta_1'''(x) = m_1\eta_1(x)\eta_1'(x) + m_2\eta_1'(x) .$$

This is the well known basic equation for the theory of solitary or cnoidal waves in channels.

For the solitary wave we impose the conditions $\eta_k(-\infty, z) = 0$ and $\phi_{ky}(-\infty, 0, z) = 0$. The latter condition guarantees that the vertical velocity is zero at $-\infty$, and from it, as we can see from (3.17) and (3.19),

$$\phi_{2y}(-\infty, 0, z) = -F_1''(-\infty) = \eta_1'(-\infty) = 0 .$$

Also, since $\phi_{2x}(-\infty, 0, 0) = c_2/\sqrt{gd}$ it follows from (3.18) that $F_2'(-\infty) = 0$ which requires $\eta_1''(-\infty)$, as (3.24) shows. With these conditions, namely

$$\eta_1(-\infty) = \eta_1'(-\infty) = \eta_1''(-\infty) = 0 ,$$

and the symmetry condition $\eta_1(x) = \eta_1(-x)$ it is not difficult to solve (3.27) and verify that

$$(3.28) \qquad \eta_1(x) = -\frac{3m_2}{m_1}\operatorname{sech}^2 \frac{x}{2}\sqrt{\frac{m_2}{m_1}} .$$

Since $m_o > 0$, we have a solitary wave only if $m_2 > 0$ and this is so only if $c_1 > 0$.

It is convenient to introduce the quantities

$$q = -\frac{3m_2\varepsilon d}{m_1} = \frac{m_2\varepsilon d}{br}$$

and

$$\ell = \sqrt{\frac{m_2\varepsilon}{m_o}} = \sqrt{\frac{brq}{dm_o}} = \sqrt{\frac{qr}{d\gamma}}$$

where

$$\gamma = \frac{m_o}{A} = \frac{\iint\limits_D (\psi_{2y}^2 + \psi_{2z}^2)\,dzdy}{\iint\limits_D dzdy}$$

and we suppose $r > 0$. In terms of these quantities, (3.28) takes the form

$$(3.29) \qquad \eta_1(x) = \frac{q}{\varepsilon d}\operatorname{sech}^2 \frac{x\ell}{2\sqrt{\varepsilon}} .$$

The pressure equation (3.4) can be written as

$$\pi(x,y,z) = \frac{c^2}{2gh} - y - \frac{\phi_x^2}{2} - \frac{1}{2\varepsilon}(\phi_y^2 + \phi_z^2)$$

and by using the foregoing results this is the same as

$$\pi(x,y,z) = \frac{\left[\sum_{k=0}^{\infty} \varepsilon^k c_k\right]^2}{2gd} - y - \frac{1}{2}\left[\sum_{k=0}^{\infty} \varepsilon^k \phi_{kx}\right]^2$$

$$- \frac{\varepsilon^3}{2}\left\{\left[\sum_{k=2}^{\infty} \varepsilon^{k-2}\phi_{ky}\right]^2 + \left[\sum_{k=2}^{\infty} \varepsilon^{k-2}\phi_{kz}\right]^2\right\}$$

or

$$\pi(x,y,z) = \frac{c_o^2}{2gd} + \frac{\varepsilon c_o c_1}{gd} - y - \frac{\phi_{ox}^2}{2} - \varepsilon\phi_{ox}\phi_{1x} + o(\varepsilon)$$

where the notation $G(\varepsilon) = o(\varepsilon)$ means

$$\lim_{\varepsilon \to 0} \frac{G(\varepsilon)}{\varepsilon} = 0 .$$

If we return to the original variables, our theory up to this point shows that:

$$f = h\eta = d\eta = d\varepsilon\eta_1 + d\varepsilon^2\eta_2 + \ldots$$

$$= q \operatorname{sech}^2 \frac{x_1 \ell}{2d} + o(q)$$

$$v_1 = u\sqrt{gh} = \phi_x\sqrt{gh} = \sqrt{gh}\,[\phi_{ox} + \varepsilon\phi_{1x}] + o(\varepsilon)$$

$$= \sqrt{gd}\left[1 + \frac{qr}{2d} - \frac{q}{d}\operatorname{sech}^2 \frac{x_1\ell}{2d}\right] + o(q)$$

$$v_2 = \frac{v\sqrt{gh}}{\sqrt{\varepsilon}} = \frac{\phi_y\sqrt{gd}}{\sqrt{\varepsilon}} = \frac{\sqrt{gd}}{\sqrt{\varepsilon}}\,[\varepsilon^2\phi_{2y} + o(\varepsilon^2)]$$

$$= -\sqrt{gd}\,\left(\frac{q}{d}\right)^{3/2}\sqrt{\frac{r}{\gamma}}\,\psi_{2y}(y,z)\operatorname{sech}^2 \frac{x_1\ell}{2d}\tanh\frac{x_1\ell}{2d} + o(q^{3/2})$$

$$v_3 = -\sqrt{gd}\,\left(\frac{q}{d}\right)^{3/2} \sqrt{\frac{r}{\gamma}}\,\psi_{2z}(y,z)\operatorname{sech}^2 \frac{x_1 \ell}{2d} \tanh \frac{x_1 \ell}{2d} + o(q^{3/2})$$

$$p = \delta g h \pi = \delta g h \left\{ \begin{matrix} \dfrac{c_o^2}{2gd} + \dfrac{\varepsilon c_o c_1}{gd} - y - \dfrac{c_o^2}{2gd} \\ - \dfrac{\varepsilon c_o}{\sqrt{gd}} [F_1'(x) + \dfrac{c_1}{\sqrt{gd}}] \end{matrix} \right\} + o(\varepsilon)$$

$$= \delta g \left[q \operatorname{sech}^2 \frac{x_1 \ell}{2d} - x_2 \right] + (q) \ .$$

In the above formulas:

q is the amplitude of the first order wave.

d is the mean depth of the liquid at infinity.

$\frac{\ell}{d} = \frac{1}{d}\sqrt{\frac{qr}{d\gamma}}$ is a scale factor which measures the sharpness of the crest of the solitary wave. If ℓ/d increases, the curvature at the crest increases, but the distance between the two points of inflexion of $\eta_1(x)$ decreases.

r and γ are quantities which depend on the cross section of the channel. They are given by

$$r = 1 - \frac{1}{3b} \int_{p_1}^{p_2} \psi_{2zz}(0,z)dz$$

$$\gamma = \frac{\iint_D (\psi_{2y}^2 + \psi_{2z}^2)dzdy}{\iint_D dzdy} \ .$$

We recall that $\psi_2(y,z)$ is defined as the solution of the following boundary value problem

$$\psi_{2yy}(y,z) + \psi_{2zz} = 1 \,, \qquad (y,z) \text{ in } D$$

$$\psi_{2y}(0,z) = 1 \,, \qquad \text{on } S$$

$$\psi_{2n} = 0 \,, \qquad (y,z) \text{ on } L$$

$$\psi_2(0,0) = 0 \,.$$

D is the domain in the zy-plane defined by $y = 0$ and $y = \omega(dz)/d$ as shown in Fig. 1. The rectangular and semicircular cross sections are examples of sections for which ψ_2 can easily be found. For the rectangle $\psi_2(y,z) = [(y+1)^2 - 1]/2$ and with this, $r = 1$ while $\gamma = 1/3$. For the semicircle we find

$$(3.30) \quad \psi_2(y,z) = \frac{z^2 + y^2}{4} + y$$

$$- \frac{a}{\pi} \operatorname{Re} \left\{ \frac{(\zeta+a)^2}{a\zeta} \ln\,(a+\zeta) - \frac{(\zeta-a)^2}{a\zeta} \ln\,(a-\zeta) - 2 - 4 \ln a \right\}$$

where $a = 4/\pi$, ζ is the complex variable $\zeta = z + iy$ and Re means the real part of the function in the brace. From (3.30) we calculate $r = 1$ and

$$\gamma = \frac{\pi}{8} \iint_D (\psi_{2y}^2 + \psi_{2z}^2)\,dz\,dy$$

$$= \frac{\pi}{8} \left[\int_{-a}^{a} \psi_2(0,z)\,dz - \iint_D \psi_2(y,z)\,dz\,dy \right]$$

$$\gamma = \frac{74 - 48 \ln 2 - 3\pi^2}{3\pi^2} \quad . \tag{3.31}$$

The velocity of the solitary wave is given by $v_1(-\infty, y, z) = c$ which is

$$c = \sqrt{gd}\,\left[1 + \frac{qr}{2d}\right] + o(q) \quad .$$

For the case of a channel of rectangular cross section, this reduces to the well known formula of Scott Russell:

$$c = \sqrt{gh}\,\left(1 + \frac{q}{2h}\right) + o(q)$$

where h is the depth.

We have summarized a theory which in some respects resembles closely that for the motion of a two-dimensional solitary wave in a rectangular channel. In fact, if the domain D is taken to be a rectangle, all of the above formulas agree with the known formulas.

Three-dimensional aspects are perhaps not very pronounced in the above approximations but they are manifest in the quantities d, ℓ, v_2 and v_3 since these depend on z. To the

order of approximation given above, the amplitude of the wave suffers no lateral change. It is interesting to note, however, that this is no longer true in the higher order approximations. The next term in the development of the wave surface depends on η_2; and we can see from (3.24) that

$$\eta_2(x,z) = -\psi_2(0,z)\eta_1''(x) + \frac{c_1\eta_1(x)}{\sqrt{gd}} - \frac{\eta_1^2(x)}{2} - F_2'(x) .$$

The term $-\psi_2(0,z)\eta_1''(x)$ modifies the amplitude with respect to the lateral z-direction. Therefore the approximation

$$x_2 = f(x_1,x_3) = d\varepsilon\eta_1\left(\frac{x_1\sqrt{\varepsilon}}{d}\right) + d\varepsilon^2\eta_2\left(\frac{x_1\sqrt{\varepsilon}}{d}, \frac{x_3}{d}\right) + o(\varepsilon^2)$$

furnishes an example of a three-dimensional solitary wave.

Notice that the sign of $-\psi_2(0,z)\eta_1''(x)$ depends on x. If s_i and $-s_i$ are the abscissas of the two points of inflexion of

$$y = \eta_1(x) = \frac{q}{\varepsilon d}\operatorname{sech}^2\frac{x\ell}{2\sqrt{\varepsilon}} , \qquad \frac{q}{\varepsilon d} > 0$$

then $\eta_1''(x) < 0$ when $|x| < |s_i|$; and $\eta_1''(x) > 0$ when $|x| > |s_i|$.

For a semicircular cross section the lateral change in amplitude from the axis $x_3 = 0$ to the side $x_3 = 4d/\pi$ is

$$f(x_1,0) - f(x_1, \tfrac{4d}{\pi}) = d\varepsilon^2\psi_2(0, \tfrac{4}{\pi})\eta_1''(x) + o(\varepsilon^2)$$

$$= \frac{2q^2}{\pi\gamma R_o}\cdot\psi_2(0, \tfrac{4}{\pi})\left[2\sinh^2\frac{x_1\ell}{2d} - 1\right]\operatorname{sech}^4\frac{x_1\ell}{2d} + o(q^2)$$

where $R_o = 4d/\pi$ is the radius of the cross section. From (3.30) we obtain

$$\psi_2(0,\frac{4}{\pi}) = \frac{4(3-4\ln 2)}{\pi^2} .$$

The ratio $\psi_2(0,\frac{4}{\pi})/\gamma$ is

$$\frac{\psi_2(0,\frac{4}{\pi})}{\gamma} = \frac{12(3-4\ln 2)}{74-3\pi^2-48\ln 2} .$$

Therefore

$$f(x_1,0) - f(x_1,\frac{4d}{\pi})$$

$$= \frac{24(3-4\ln 2)}{\pi(74-3\pi^2-48\ln 2)}\cdot\frac{q^2}{R_o}\cdot\left[2\sinh^2\frac{x_1\ell}{d}-1\right]\operatorname{sech}^4\frac{x_1\ell}{2d} + o(q^2) .$$

At $x_1 = 0$ this is approximately equal to

$$f(0,0) - f(0,\frac{4d}{\pi}) = -(.156)\,\frac{q^2}{R_o} .$$

4. A Rotational Solitary Wave in a Channel with Arbitrary Cross Section

In 1953 Hunt [10] studied the problem of two-dimensional rotational solitary and cnoidal waves generated by applying a disturbance to an inviscid liquid stream of constant density with vorticity depending continuously on the depth. Assuming the existence of such waves Hunt showed how they could be

approximated. Subsequently, the problem was investigated in more detail within the framework of perturbation theory by Moiseev [11] (1960), Ter-Krikorov [12] (1961); and Benjamin [13] (1962). Ter-Krikorov proved the existence of the two-dimensional rotational wave by using the method which Friedrichs and Hyers [6] devised for the irrotational case. In 1959 Peters and Stoker [3] presented an analysis of a two-dimensional solitary wave with a continuous density stratification — another kind of rotational wave. Since the wave analyzed in [3] is assumed to arise from the rest position of the non-homogeneous liquid its vorticity is self-generated. Shen [14] (1965) extended the results of [3] and [13] to two-dimensional rotational waves in compressible media. None of the methods in these papers can be extended to three dimensions because each depends essentially on the use of a stream function peculiar to two dimensions. In order to discuss the possibility of three-dimensional rotational solitary and cnoidal waves we must return to the basic equations (2.1)-(2.4).

The purpose of this section is to show that the variant of Friedrichs' technique which leads to equations (2.5)-(2.8), plus some auxiliary devices, can be used to investigate rotational solitary waves in a stream of constant density with an arbitrary lateral distribution of velocity, $v_o(y,z)$, (independent of x) when the stream is confined to a channel of arbitrary cross section. We assume that a coordinate system is chosen as it was in Section 3; and we suppose that the vertical scale

factor is taken equal to the mean depth of the liquid at infinity. That is, as before,

$$h = d .$$

With respect to the above coordinate system, the basic equations are:

$$\varepsilon \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.1}$$

$$\left\{\begin{aligned} &\varepsilon[u \frac{\partial u}{\partial x} + \frac{\partial \pi}{\partial x}] + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 0 \\ &\varepsilon[u \frac{\partial v}{\partial x} + 1 + \frac{\partial \pi}{\partial y}] + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = 0 \\ &\varepsilon[u \frac{\partial w}{\partial x} + \frac{\partial \pi}{\partial z}] + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = 0 \end{aligned}\right. \tag{4.2}$$

$$v = w \frac{\partial \omega}{\partial z} \qquad \text{for} \qquad y = \omega(z) \tag{4.3}$$

$$\varepsilon u \frac{\partial \eta}{\partial x} = v - w \frac{\partial \eta}{\partial z} \qquad \text{for} \qquad y = \eta(x,z) \tag{4.4}$$

$$\varepsilon \pi(x,\eta,z) = 0 . \tag{4.5}$$

The idea once again is to develop the dependent quantities in integral powers of ε and then solve the partial differential systems which result from equating coefficients of like powers of ε.

The horizontal velocity component u — the axial velocity — is to be regarded as a sum of two parts

$$u = \frac{v_o(y,z)}{\sqrt{gd}} + U(x,y,z) \ .$$

The velocity

$$V_o(x_2,x_3) = V_o(dy,dz) = v_o(y,z) \ ,$$

which we suppose is continuous and non-negative, is the velocity of the undisturbed liquid stream which has no lateral velocity components, and which is confined to a horizontal channel of infinite length with arbitrary cross section. We assume that the velocity U due to a steady state solitary wave can be expanded in the form

$$U = \frac{c_o}{\sqrt{gd}} + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \quad .$$

We can thus suppose that

$$u = u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

where

$$u_o = \frac{c_o + v_o(y,z)}{\sqrt{gd}} \ . \tag{4.6}$$

Guided by what we have found for the irrotational case we also assume that

$$v = \varepsilon^2 \nu_2 + \varepsilon^3 \nu_3 + \dots$$

$$w = \varepsilon^2 \mu_2 + \varepsilon^3 \mu_3 + \dots$$

$$\pi = -y + \varepsilon \pi_1 + \varepsilon^2 \pi_2 + \dots$$

$$\eta = \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \dots \quad .$$

If we substitute the above expansions in (4.1), (4.2), (4.3), (4.4), (4.5) and equate coefficients of ε^2 in each equation we find the system:

$$u_{1x} + \nu_{2y} + \mu_{2z} = 0 \tag{4.7}$$

$$u_o u_{1x} + \pi_{1x} + \nu_2 u_{oy} + \mu_2 u_{oz} = 0 \tag{4.8}$$

$$\pi_{1y} = 0 \tag{4.9}$$

$$\pi_{1z} = 0 \tag{4.10}$$

$$\nu_2 = \mu_2 \frac{\partial\omega}{\partial z}, \qquad y = \omega(z) \tag{4.11}$$

$$u_o(0,z)\eta_{1x}(x,z) = \nu_2(x,0,z) \tag{4.12}$$

$$\pi_1(x,0,z) = \eta_1(x,z) . \tag{4.13}$$

Equations (4.9), (4.10) and (4.13) show that

$$\pi_1(x,y,z) = \eta_1(x)$$

where $\eta_1(x)$ is a function of x only. The elimination of u_1 from (4.7) and (4.8) gives

$$\frac{\partial}{\partial y}\left(\frac{\nu_2}{u_o}\right) + \frac{\partial}{\partial z}\left(\frac{\mu_2}{u_o}\right) = \frac{\eta_{1x}}{u_o^2} . \tag{4.14}$$

Let $\underline{i}$ and $\underline{j}$ be unit vectors along the z-axis and y-axis respectively. Let $\underline{V}$ be the vector

$$V = \mu \underline{i} + \nu \underline{j}$$

and let

$$\nabla = \underline{i}\frac{\partial}{\partial z} + \underline{j}\frac{\partial}{\partial y} .$$

If $\underline{n}$ is the unit outward normal of the domain D defined in Section 3, the divergence theorem states

$$\iint_D \nabla\cdot\underline{V}\, dzdy = \int_{L+S} \underline{V}\cdot\underline{n}\, ds . \tag{4.15}$$

Along L we have

$$(\mu_2\underline{i} + \nu_2\underline{j})\cdot\underline{n} = 0$$

as we can see from (4.11). Hence if we apply (4.15) to the vector

$$\frac{\mu_2\underline{i} + \nu_2\underline{j}}{u_o}$$

the equation (4.14) implies

$$\eta_{1x} \iint_D \frac{dzdy}{u_o^2} = \int_{p_1}^{p_2} \frac{\nu_2(x,0,z)dz}{u_o}$$

and then from (4.12) we have the condition

$$\eta_{1x}(x)\left[\iint_D \frac{dzdy}{u_o^2} - b\right] = 0 .$$

Here again we have a bifurcation phenomenon. Either $\eta_{1x} = 0$ or else

$$\iint_D \frac{dzdy}{u_o^2} = b .$$

The first alternative is uninteresting since it allows only the equilibrium flow. The second alternative is an equation for the determination of c_o, namely,

$$\iint_D \frac{dzdy}{[c_o + v_o(z,y)]^2} = \frac{b}{gd} . \tag{4.17}$$

Equation (4.17) specifies two values of c_o. This can be seen by tracing the change in the integral

$$I(c_o) = \iint_D \frac{dzdy}{[c_o + v_o(z,y)]^2}$$

as c_o varies from $-\infty$ to ∞. We suppose that $v_o(z,y)$ is a non-negative continuous function on the closure of D, and that m and M are respectively its minimum and maximum values there. We cannot admit values of c_o between -M and -m because for such values $I(c_o)$ fails to exist. If c_o varies from -m to $+\infty$, $I(c_o)$ changes from $+\infty$ to zero; furthermore as we can see from

$$I'(c_o) = -2 \iint_D \frac{dzdy}{[c_o + v_o(z,y)]^3}$$

it never increases in the range $-m < c_o$. Hence there is one value $c_o > -m$ which satisfies (4.17); and this value corresponds to a wave advancing upstream. If c_o varies from $-\infty$ to -M, $I(c_o)$ changes from zero to $+\infty$; and in the range $-\infty < c_o < -M$, $I'(c_o) \geq 0$. Thus there is one value of $c_o < -M$ which satisfies (4.17) and it corresponds to a wave progressing downstream.

These values of c_o will be referred to as critical values. More can be said about equation (4.17) and we will continue the discussion in the next section.

In order to find the dependent quantities u_1, μ_2, ν_2 and η_1 we turn to the next partial differential system which comes from equating coefficients of ε^2 in each of the expanded equations (4.1)-(4.5). It is:

$$u_{2x} + \nu_{3y} + \mu_{3z} = 0 \tag{4.18}$$

$$u_o u_{2x} + u_1 u_{1x} + \pi_{2x} + \nu_2 u_{1y} + \nu_3 u_{oy} + \mu_2 u_{1z} + \mu_3 u_{oz} = 0 \tag{4.19}$$

$$u_o \nu_{2x} + \pi_{2y} = 0 \tag{4.20}$$

$$u_o \mu_{2x} + \pi_{2z} = 0 \tag{4.21}$$

$$\nu_3 = \mu_3 \omega'(z) \ , \qquad y = \omega(z) \tag{4.22}$$

$$\begin{aligned} u_{oy}(0,z)\eta_1\eta_{1x} + u_1(x,0,z)\eta_{1x} + u_o(0,z)\eta_{2x} \\ = \nu_{2y}(x,0,z)\eta_1 + \nu_3(x,0,z) \end{aligned} \tag{4.23}$$

$$\pi_2(x,0,z) = \eta_2(x,z) \ . \tag{4.24}$$

The differentiation of (4.14) with respect to x gives

$$(4.25) \qquad \frac{\partial}{\partial y}\left(\frac{\nu_{2x}}{u_o}\right) + \frac{\partial}{\partial z}\left(\frac{\mu_{2x}}{u_o}\right) = \frac{\eta_{1xx}}{u_o^2}$$

and if we use (4.20) and (4.21) the last equation becomes

$$(4.26) \qquad \frac{\partial}{\partial y}\left(\frac{\pi_{2y}}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{\pi_{2z}}{u_o^2}\right) = -\frac{\eta_{1xx}}{u_o^2} .$$

We also have from (4.20) and (4.12) that

$$(4.27) \qquad \pi_{2y}(x,0,z) = -u_o\nu_{2x}(x,0,z) = -u_o^2(0,z)\eta_{1xx}(x) .$$

In addition, (4.11) shows that

$$\nu_{2x} = \mu_{2x}\omega'(z)$$

$$u_o\nu_{2x} = u_o\mu_{2x}\omega'(z)$$

or

$$\pi_{2y} = \pi_{2z}\omega'(z)$$

which means that the normal derivative π_{2n} must satisfy

$$(4.28) \qquad \pi_{2n} = 0$$

along L.

The solution of the boundary value problem posed by (4.26), (4.27) and (4.28) can be expressed in the form

$$(4.29) \qquad \pi_2(x,y,z) = -\psi(z,y)\eta_{1xx} + F(x) .$$

Here $\psi(z,y)$ must satisfy

$$(4.30) \qquad \frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right) = \frac{1}{u_o^2}$$

for (y,z) in D. This is an elliptic partial differential equation, but not Laplace's equation, since u_o, as given by (4.6), is a known function of y and z. The boundary conditions for ψ are

$$(4.31) \qquad \psi_y(0,z) = u_o^2(0,z)$$

for the segment S, while along L

$$(4.32) \qquad \psi_n = 0 \ .$$

We will make ψ unique by imposing (without loss of generality) some convenient additional condition at a point, say

$$(4.33) \qquad \psi(0,0) = 0 \ .$$

The functions ν_2, μ_2 and u_1 can now be expressed in terms of ψ and η_1. From (4.20) and (4.21) we have

$$u_o \nu_{2x} = -\pi_{2y} = \psi_y \eta_{1xx}$$

$$u_o \mu_{2x} = -\pi_{2z} = \psi_z \eta_{1xx}$$

and integration with respect to x gives

$$u_o \nu_2 = \psi_y \eta_{1x} + R_1(y,z)$$

$$u_o \mu_2 = \psi_z \eta_{1x} + R_2(y,z) \ .$$

Since we are interested here primarily in the solitary wave we impose the conditions

$$\eta_1(\infty) = 0 , \quad u_1(\infty, y, z) = c_1$$

(4.34)

$$\nu_2(\infty, y, z) = \mu_2(\infty, y, z) = 0 .$$

If $\nu_2(\infty, 0, z) = 0$ equation (4.12) requires $\eta_{1x}(\infty) = 0$ and consequently

(4.35) $$\nu_2(x,y,z) = \frac{\psi_y(y,z)}{u_o(y,z)} \eta_{1x}(x)$$

(4.36) $$\mu_2(x,y,z) = \frac{\psi_z(y,z)}{u_o(y,z)} \eta_{1x}(x) .$$

For the function u_1, equation (4.7) requires

$$u_{1x} = -\nu_{2y} - \mu_{2z} = -\left[\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o}\right)\right]\eta_{1x}$$

and after integration with respect to x and use of (4.34)

(4.37) $$u_1 = c_1 - \left[\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o}\right)\right]\eta_1 .$$

If we define functions P and Q by

(4.38) $$P(y,z) = \frac{\psi_y(y,z)}{u_o(y,z)} , \quad Q(y,z) = \frac{\psi_z(y,z)}{u_o(y,z)}$$

we have

$$(4.39) \qquad \nu_2(x,y,z) = P(y,z)\eta_{1x}(x)$$

$$(4.40) \qquad \mu_2(x,y,z) = Q(y,z)\eta_{1x}(x)$$

$$(4.41) \qquad u_1(x,y,z) = c_1 - [P_y(y,z) + Q_z(y,z)]\eta_1(x) \; .$$

To this point we have not used the equations (4.18), (4.19), (4.23) and (4.24). We proceed to show that these equations serve for the determination of $\eta_1(x)$. The elimination of u_2 from (4.18) and (4.19) leads to an equation which, after some manipulation, can be written in the convenient form

$$(4.42) \qquad \frac{\partial}{\partial y}\left[\frac{\nu_3}{u_o} - \frac{\nu_2 u_1}{u_o^2}\right] + \frac{\partial}{\partial z}\left[\frac{\mu_3}{u_o} - \frac{\mu_2 u_1}{u_o^2}\right] = \frac{\pi_{2x}}{u_o^2} - \frac{2u_1\pi_{1x}}{u_o^3} \; .$$

If we set

$$\underline{V} = \frac{[u_o\mu_3 - \mu_2 u_1]}{u_o^2}\,\underline{i} + \frac{[u_o\nu_3 - \nu_2 u_1]}{u_o^2}\,\underline{j}$$

and apply the divergence theorem, (4.15), to (4.42) we find

$$(4.43) \qquad \iint\limits_D \left[\frac{\pi_{2x}}{u_o^2} - \frac{2u_1\pi_{1x}}{u_o^3}\right] dzdy = \int_{p_1}^{p_2} \frac{[u_o\nu_3 - \nu_2 u_1]}{u_o^2}\, dz \; .$$

The boundary conditions (4.23), (4.24) and (4.12) show that (4.43) is the same as

$$(4.44)\quad \iint_D \left[\frac{\pi_{2x}}{u_o^2} - \frac{2u_1\pi_{1x}}{u_o^3}\right] dzdy$$

$$= \int_{p_1}^{p_2} \frac{[u_{oy}(0,z)\eta_1\eta_{1x} + u_o(0,z)\pi_{2x}(x,0,z) - \nu_{2y}(x,0,z)\eta_1]}{u_o(0,z)}\, dz\ .$$

Then, after using the results (4.29), (4.39), (4.41) and $\pi_1 = \eta_1(x)$, we have

$$(4.45)\quad \iint_D \left\{\frac{-\psi\eta_{1xxx} + F'}{u_o^2} - \frac{2[c_1 - (P_y + Q_z)\eta_1]\eta_{1z}}{u_o^3}\right\} dzdy$$

$$= \int_{p_1}^{p_2} \left\{\frac{u_{oy}(0,z)\eta_1\eta_{1x}}{u_o(0,z)} - \psi\eta_{1xxx} + F' - \frac{P_y(0,z)\eta_1\eta_{1x}}{u_o(0,z)}\right\} dz\ .$$

The coefficient of $F'(x)$ vanishes by virtue of (4.16) and therefore (4.45) reduces to

$$(4.46)\quad \left\{\int_{p_1}^{p_2} \psi(0,z)dz - \iint_D \frac{\psi(z,y)}{u_o^2}\, dzdy\right\}\eta_{1xxx}$$

$$= \left\{\int_{p_1}^{p_2} \frac{[u_{oy}(0,z) - P_y(0,z)]}{u_o(0,z)}\, dz - \iint_D \frac{2(P_y + Q_z)}{u_o^3}\, dzdy\right\}\eta_1\eta_{1x}$$

$$+ \left\{2c_1 \iint_D \frac{dzdy}{u_o^3}\right\}\eta_{1x}$$

which is again the familiar ordinary non-linear differential

equation for the determination of the first order approximation to the surface wave $\eta(x,z)$ of solitary type.

The coefficient of η_{1xxx} can be transformed as follows.

$$m_o = \int_{p_1}^{p_2} \psi(0,z)dz - \iint_D \frac{\psi(z,y)}{u_o^2} dzdy$$

$$= \int_{p_1}^{p_2} \psi(0,z)dz - \iint_D \psi\left\{\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right)\right\} dzdy$$

$$= \int_{p_1}^{p_2} \psi(0,z)dz - \int_{p_1}^{p_2} \psi(0,z)\, \frac{\psi_y(0,z)}{u_o^2(0,z)}\, dz + \iint_D \frac{(\psi_y^2 + \psi_z^2)}{u_o^2}\, dzdy \ .$$

From (4.31), namely $\psi_{2y}(0,z) = u_o^2(0,z)$ we see that

$$m_o = \iint_D \frac{(\psi_y^2 + \psi_z^2)}{u_o^2}\, dzdy$$

which shows that m_o is always positive. With respect to the coefficient of $\eta_1\eta_{1x}$ notice first that

$$P_y + Q_z = u_o\left\{\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right)\right\} + \frac{u_{oy}\psi_y + u_{oz}\psi_z}{u_o^2}$$

$$= \frac{1}{u_o} + \frac{u_{oy}\psi_y + u_{oz}\psi_z}{u_o^2} \quad .$$

Then

$$m_1 = \int_{p_1}^{p_2} \left\{ \frac{u_{oy}(0,z)}{u_o(0,z)} - \frac{\psi_{yy}(0,z)}{u_o^2(0,z)} + \frac{u_{oy}(0,z)\psi_y(0,z)}{u_o^3(0,z)} \right\} dz$$

$$- 2 \iint_D \left\{ \frac{1}{u_o^4} + \frac{u_{oy}\psi_y + u_{oz}\psi_z}{u_o^5} \right\} dzdy$$

$$= \int_{p_1}^{p_2} \left[\frac{2u_{oy}(0,z)}{u_o(0,z)} - \frac{\psi_{yy}(0,z)}{u_o^2(0,z)} \right] dz - 2 \iint_D \frac{dzdy}{u_o^4}$$

$$+ \int_{p_1}^{p_2} \frac{\psi_y(0,z)}{u_o^4(0,z)} \, dz - \iint_D \frac{1}{u_o^2} \left[\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right) \right] dzdy$$

$$= \int_{p_1}^{p_2} \left[\frac{2u_{oy}(0,z)\psi_y(0,z)}{u_o^3(0,z)} - \frac{\psi_{yy}(0,z)}{u_o^2(0,z)} + \frac{1}{u_o^2(0,z)} \right] dz - 3 \iint_D \frac{dzdy}{u_o^4}$$

$$m_1 = \int_{p_1}^{p_2} \frac{\partial}{\partial z} \left[\frac{\psi_z(0,z)}{u_o^2(0,z)} \right] dz - 3 \iint_D \frac{dzdy}{u_o^4} \quad .$$

Note that if the free surface of the liquid at infinity intersects the channel wall orthogonally then

$$\int_{p_1}^{p_2} \frac{\partial}{\partial z} \left[\frac{\psi_z(0,z)}{u_o^2(0,z)} \right] dz = 0 \; .$$

The coefficient of η_{1x} will be denoted by m_2, that is,

$$m_2 = 2c_1 \iint_D \frac{dzdy}{u_o^3} \; .$$

In terms of the coefficients just introduced the equation (4.46) is

$$m_0\eta_1'''(x) = m_1\eta_1(x)\eta_1'(x) + m_2\eta_1'(x) \quad . \tag{4.47}$$

For the solitary wave we impose the conditions

$$\begin{aligned} &\eta_k(\infty,z) = 0 \quad , \quad u_k(\infty,y,z) = c_k \\ &\nu_k(\infty,y,z) = 0 \ , \quad \mu_k(\infty,y,z) = 0 \quad . \end{aligned} \tag{4.48}$$

As we have already seen these conditions imply $\eta_1(\infty) = \eta_{1x}(\infty) = 0$
From (4.24) and (4.29) the second order profile $\eta_2(x,z)$ is

$$\eta_2(x,z) = \pi_2(x,0,z) = -\psi(0,z)\eta_{1xx}(x) + F(x) \quad .$$

In order to satisfy

$$0 = \eta_2(\infty,z) = -\psi(0,z)\eta_{1xx}(\infty) + F(\infty)$$

we take $F(\infty) = 0$ and $\eta_{1xx}(\infty) = 0$. Subject to the conditions

$$\eta_1''(\infty) = \eta_1'(\infty) = \eta_1(\infty) = 0$$

and the symmetry condition

$$\eta_1(-x) = \eta_1(x) \quad ,$$

the solution of (4.47) is

$$\eta_1(x) = -\frac{3m_2}{m_1}\,\mathrm{sech}^2\,\frac{x}{2}\sqrt{\frac{m_2}{m_0}} \tag{4.49}$$

provided c_1 is chosen so that $m_2 > 0$.

For later use it is convenient to define

$$q = -\frac{3m_2\varepsilon d}{m_1}$$

and

$$\ell = \sqrt{\frac{m_2\varepsilon}{m_o}} = \sqrt{\frac{q}{d}\left(\frac{-m_1}{3m_o}\right)} = \sqrt{\frac{q\kappa}{dm_o}}$$

where

$$\kappa = \iint_D \frac{dzdy}{u_o^4} - \frac{1}{3}\int_{p_1}^{p_2} \frac{\partial}{\partial z}\left[\frac{\psi_z(0,z)}{u_o^2(0,z)}\right] dz \quad .$$

In terms of these quantities (4.49) is

$$\eta_1 = \frac{q}{\varepsilon d}\,\text{sech}^2\,\frac{x\ell}{2\sqrt{\varepsilon}} \quad . \tag{4.50}$$

We could go on to higher order approximations but we do not think that this is the place to show how the ensuing complications can be dealt with. Instead, it is more to the point to collect the results we have found and express them in terms of the original variables with respect to a coordinate system moving with the wave. We have shown that the free surface is given by

$$f = d\eta = d\varepsilon\eta_1 + o(\varepsilon)$$

$$= q\,\text{sech}^2\,\frac{x_1\ell}{2d} + o(q) \quad .$$

The long itudinal velocity has the form

$$v_1 = \sqrt{gd}\left[\frac{v_o(y,z)+c_o}{\sqrt{gd}} + \varepsilon u_1\right] + o(\varepsilon)$$

$$= V_o(x_2,x_3) + c_o + \sqrt{\frac{g}{d}}\cdot\frac{\kappa q}{2\iint_D \frac{dzdy}{u_o^3}}$$

$$- q\sqrt{\frac{g}{d}}\left\{\frac{1}{u_o} + \frac{u_{oy}\psi_y + u_{oz}\psi_z}{u_o^2}\right\}\operatorname{sech}^2\frac{x_1\ell}{2d} + o(q) \ .$$

The lateral velocity components are given by

$$v_2 = \sqrt{\frac{gd}{\varepsilon}}\, v = \sqrt{\frac{gd}{\varepsilon}}\,[\varepsilon^2 \nu_2 + o(\varepsilon^2)]$$

$$= -\sqrt{gd}\left(\frac{q}{d}\right)^{3/2}\left(\frac{\kappa}{m_o}\right)^{1/2}\frac{\psi_y(y,z)}{u_o(y,z)}\operatorname{sech}^2\frac{x_1\ell}{2d}\cdot\tanh\frac{x_1\ell}{2d} + o(q^{3/2})$$

and

$$v_3 = -\sqrt{gd}\left(\frac{q}{d}\right)^{3/2}\left(\frac{\kappa}{m_o}\right)^{1/2}\frac{\psi_z(y,z)}{u_o(y,z)}\operatorname{sech}^2\frac{x_1\ell}{2d}\cdot\tanh\frac{x_1\ell}{2d} + o(q^{3/2}) \ .$$

The pressure p is given by

$$p = \delta g d\pi = \delta g d\,[-y + \varepsilon\pi_1 + o(\varepsilon)]$$

$$= \delta g\,[q\operatorname{sech}^2\frac{x_1\ell}{2d} - x_2] + o(q)$$

and hence to this order of approximation the pressure obeys the hydrostatic law.

These formulas summarize the results of this section and in fact subsume the results of the last section.

For ready reference we append a table of definitions of the terms, other than those with obvious meanings, in the above formulas:

q is the amplitude of the first order wave.

d is the mean depth of the liquid at infinity.

$\frac{\ell}{d} = \frac{1}{d}\sqrt{\frac{q\kappa}{dm_o}}$ is a scale factor which is a measure of the sharpness of the crest of the solitary wave. If $\frac{\ell}{d}$ increases the curvature at the crest increases, but the distance between the two points of inflexion of the solitary wave decreases.

$$\kappa = \iint_D \frac{dzdy}{u_o^4} - \frac{1}{3}\int_{p_1}^{p_2} \frac{\partial}{\partial z}\left[\frac{\psi_z(0,z)}{u_o^2(0,z)}\right] dz$$

$$m_o = \iint_D \frac{[\psi_y^2 + \psi_z^2]}{u_o^2}\, dzdy$$

$\psi(y,z)$ is defined by

$$\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right) = \frac{1}{u_o^2} \qquad (y,z) \text{ in } D$$

$$\psi_y(0,z) = u_o^2(0,z)$$

$$\psi_n = 0 \qquad (y,z) \text{ on } L$$

$$\psi(0,0) = 0 \quad .$$

D is the domain in the zy-plane defined by $y = 0$ and $y = \omega(z) = W(dz)/d$ as shown in Fig. 1.

$$u_0 = \frac{c_0 + v_0(y,z)}{\sqrt{gd}} = \frac{c_0 + V_0(dy,dz)}{\sqrt{gd}} = \frac{c_0 + V_0(x_2,x_3)}{\sqrt{gd}}$$

where $V_0(x_2,x_3)$ is the longitudinal velocity of the undisturbed liquid stream.

c_0 is a critical speed specified by

$$\iint_D \frac{dzdy}{[c_0 + v_0(z,y)]^2} = \frac{b}{gd}$$

where bd is the breadth of the free surface at infinity.

The velocity c of the solitary wave is given by

$$-c = [v_1(-\infty,y,z) - v_0(y,z)] \ ,$$

that is,

$$-c = c_0 + \frac{\kappa q \sqrt{\frac{g}{d}}}{2\iint_D \frac{dzdy}{u_0^3}} + o(q) \ . \tag{4.51}$$

5. Discussion

Three dimensional aspects in the approximations given in Section 4 are manifest in the quantities d, ℓ, v_1, v_2 and v_3; but to the order of approximation given above, the amplitude of the wave suffers no lateral change, i.e. no change in the z-direction. It is interesting to observe however, that the higher order approximations do show changes in amplitude in the z-direction. The next term in the development of the wave surface depends on η_2; and we can see from (4.24) and (4.29) that

$$\eta_2(x,z) = \pi_2(x,0,z) = -\psi(0,z)\eta_{1xx}(x) + F(x) .$$

The term $-\psi(0,z)\eta_{1xx}(x)$, whose sign depends on x, modifies the amplitude with respect to the lateral z-direction. Hence the approximation

$$f = d\varepsilon\eta_1 + d\varepsilon^2\eta_2 + o(\varepsilon^2)$$

evidently represents a three-dimensional solitary wave.

The linear theory of surface waves on a liquid stream in a channel is based on the equations which can be obtained from (4.1)-(4.5) by substituting $\varepsilon = 1$, $u = u_o + u^*$, $v = v^*$, $w = w^*$, $\pi = -y + \pi^*$; $\eta = \eta^*$ and then neglecting products of starred quantities. The equations are:

(5.1) $$u_x^* + v_y^* + w_z^* = 0$$

(5.2) $$\begin{cases} u_o u_x^* + v^* u_{oy} + w^* u_{oz} = -\pi_x^* \\ u_o v_x^* = -\pi_y^* \\ u_o w_x^* = -\pi_z^* \end{cases}$$

(5.3) $$v^* = w^*\omega'(z) \quad \text{for} \quad y = \omega(z)$$

(5.4) $$u_o\eta_x^* = v^* \quad \text{for} \quad y = 0$$

(5.5) $$-\eta^* + \pi^* = 0 \quad \text{for} \quad y = 0 .$$

If we eliminate u^*, v^*, w^*, and η^* we find that π^* must satisfy

$$\frac{\partial}{\partial x}\left(\frac{\pi_x^*}{u_o^2}\right) + \frac{\partial}{\partial y}\left(\frac{\pi_y^*}{u_o^2}\right) + \frac{\partial}{\partial z}\ \frac{\pi_z^*}{u_o^2} = 0$$

for (y,z) in D and the boundary conditions

$$u_o^2(0,z)\pi_{xx}^*(x,0,z) + \pi_y^*(x,0,z) = 0$$

for S and

$$\pi_n^* = 0$$

for L. For solutions of the form

$$\pi^* = \psi^*(y,z)e^{i\nu x} \tag{5.6}$$

we must have

$$(5.7)\qquad \begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\psi_y^*}{u_o^2}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\psi_z^*}{u_o^2}\right) = \dfrac{\nu^2}{u_o^2}\,\psi^* & (y,z) \text{ in } D \\[2ex] \psi_y^*(0,z) = \nu^2 u_o^2(0,z)\psi^*(0,z) & \\[2ex] \psi_n^* = 0 & (y,z) \text{ on } L\ . \end{cases}$$

The divergence theorem applied to (5.7) yields

$$\iint_D \frac{\psi^*}{u_o^2}\,dzdy = \int_{p_1}^{p_2} \psi^*(0,z)dz\ . \tag{5.8}$$

As $\nu \longrightarrow 0$ the wave length of (5.6) $\longrightarrow \infty$; but $\psi^* \longrightarrow$ constant. The condition (5.8) then becomes

$$(5.9) \qquad \iint_D \frac{dzdy}{u_o^2} = \iint_D \frac{(gd)dzdy}{[c_o + v_o(z,y)]^2} = b$$

which we can expect to yield the approximate speeds of long waves progressing against and with the stream. Equation (5.9) is of course precisely (4.17) which gives the approximate speeds of the solitary wave. For a rectangular channel (5.9) becomes

$$(5.10) \qquad \int_{-1}^{0} \frac{dy}{u_o^2} = 1$$

or

$$(5.11) \qquad \int_{-1}^{0} \frac{dy}{[c_o + v_o(y)]^2} = \frac{1}{gd} .$$

The last equation was derived by Burns [15] from a two-dimensional analysis based on ideas different from those above. Burns gave a method for estimating the speed of long two-dimensional waves relative to the mean of the velocity $v_o(y)$. His method is also applicable to the equation (5.9) as we proceed to show.

If the mean flow is

$$\bar{v}_o = \frac{1}{A} \iint_D v_o(z,y)dzdy$$

then

$$\frac{\bar{v}_o + c_o}{\sqrt{gd}} = \frac{1}{A} \iint_D \frac{(v_o + c_o)}{\sqrt{gd}} dzdy = \frac{1}{A} \iint_D u_o dzdy .$$

Also since $b/A = 1$, (5.9) is the same as

$$\iint_D \frac{dzdy}{u_o^2} = A$$

and therefore

$$\frac{(\overline{v}_o + c_o)^2}{gd} = \frac{1}{A^2}\left[\iint_D u_o dzdy\right]^2$$

$$= \frac{1}{A^3}\left[\iint_D u_o dzdy\right]^2 \iint_D \frac{dzdy}{u_o^2}$$

$$= \frac{1}{A^3}\left[\iint_D |u_o| dzdy\right]^2 \iint_D \frac{dzdy}{|u_o|^2} \quad .$$

Successive applications of Schwartz' inequality produces

$$\frac{(\overline{v}_o + c_o)^2}{gd} \geq \frac{1}{A^3} \iint_D |u_o| dzdy \left[\iint_D \frac{dzdy}{|u_o|^{1/2}}\right]^2$$

$$\geq \frac{1}{A^3} \iint_D \frac{dzdy}{|u_o|^{1/2}} \cdot \left[\iint_D |u_o|^{1/4}\, dzdy\right]^2$$

$$\geq \frac{1}{A^3} \iint_D |u_o|^{1/4}\, dzdy \cdot \left[\iint_D \frac{dzdy}{|u_o|^{1/8}}\right]^2$$

$$\geq \frac{1}{A^3} \iint_D \frac{dzdy}{|u_o|^{1/8}} \left[\iint_D |u_o|^{1/16}\, dzdy\right]^2$$

$$\geq \frac{1}{A^3} \iint_D |u_o|^{1/16}\, dzdy \left[\iint_D \frac{dzdy}{|u_o|^{1/32}}\right]^2$$

$$\cdots\cdots\cdots\cdots$$

$$\geq \frac{1}{A^3} \iint_D |u_o|^{1/2p}\, dzdy \left[\iint_D \frac{dzdy}{|u_o|^{1/2^{2p+1}}}\right]^2 \quad .$$

Now

$$\lim_{n \to \infty} |x|^{1/n} = 1 .$$

Hence

$$|\bar{v}_o + c_o| \geq \sqrt{gd} \quad . \tag{5.12}$$

That is, the magnitude of the critical speed with respect to the mean velocity of the flow is never less than the critical velocity of the flow without vorticity. The same thing holds for the first order wave velocity as we can see from

$$\bar{v}_o - c = \bar{v}_o + c_o + \frac{\kappa q \sqrt{\frac{g}{d}}}{2 \iint_D \frac{dzdy}{u_o^3}} + o(q) .$$

If $c_o > -m$, $c_o + m > 0$ then

$$\bar{v}_o + c_o > m + c_o > 0$$

$$c_o + v_o > c_o + m > 0$$

$$\kappa > 0$$

$$2 \iint_D \frac{dzdy}{u_o^3} > 0$$

and therefore

$$\bar{v}_o - c > \sqrt{gd} .$$

If $c_o < -M$:

$$\bar{v}_o - c_o > 0$$

$$\bar{v}_o - c_o \geq \sqrt{gd}$$

$$\overline{v}_o + c = \overline{v}_o - c_o - \frac{\kappa q \sqrt{\frac{g}{d}}}{2 \iint \frac{dzdy}{u_o^3}} + o(q)$$

$$c_o + v_o < -M + v_o < 0$$

$$2 \iint\limits_D \frac{dzdy}{u_o^3} < 0$$

$$\kappa > 0$$

and therefore

$$\overline{v}_o + c > \sqrt{gd} \quad .$$

If the amplitude q and the channel section D are given, the actual computations of the wave profile, particle velocity and pressure depend on the evaluation of quantities which involve the function ψ. This is a considerable task and it is even more difficult to estimate the effects of vorticity by comparing parameters in the rotational wave with the corresponding parameters in the irrotational wave which depend on ψ_2. We do no three-dimensional computation here. Instead, we will devote a little attention to the more tractable case of the two-dimensional rotational wave.

For a channel with a rectangular cross section of depth h containing a stream whose velocity $V_o(x_2,x_3) = V_o(hy,hz) = v_o(y,z)$ is a function of y alone, i.e., $v_o = v_o(y)$, the wave profile is given by

$$f = q \operatorname{sech}^2 \frac{x_1}{2h} \sqrt{\frac{\kappa q}{hm_o}} \tag{5.13}$$

where

$$\kappa = \iint_D \frac{dzdy}{u_o^4} = b(gd)^2 \int_{-1}^{0} \frac{dy}{[c_o + v_o(y)]^2}$$

and

$$m_o = \iint_D \frac{\psi_y^2 dzdy}{u_o^2} = bgd \int_{-1}^{0} \frac{\psi_y^2 dy}{[c_o + v_o(y)]^2} .$$

The function ψ is defined by

$$\frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) = \frac{1}{u_o^2} , \qquad -1 < y < 0$$

$$\psi_y(0) = u_o^2(0)$$

$$\psi_y(-1) = 0$$

$$\psi(0) = 0 .$$

Integration gives

$$\frac{\psi_y(y)}{u_o^2(y)} = \int_{-1}^{y} \frac{d\xi}{u_o^2(\xi)}$$

and so

$$m_o = bgd \int_{-1}^{0} [c_o + v_o(y)]^2 \left\{ \int_{-1}^{y} \frac{d\xi}{[c_o + v_o(\xi)]^2} \right\}^2 dy .$$

Benjamin's result [13] can be transformed into

$$f = q \operatorname{sech}^2 \frac{x_1}{2h} \sqrt{\frac{qK_o}{hM_o}} \tag{5.14}$$

where

$$K_0 = b(gd)^2 \int_{-1}^{0} \frac{dy}{[v_0(y) - c]^4}$$

and

$$M_0 = bgd \int_{-1}^{0} [v_0(y) - c]^2 \left\{ \int_{-1}^{y} \frac{d\xi}{[v_0(\xi) - c]^2} \right\}^2 dy \ .$$

Here c denotes the actual velocity of the wave. However, as we can see from (4.51),

$$c_0 + c = O(1)$$

which means that (5.13) and (5.14) give the same first order approximation. Our result (5.13) is also in essential agreement with the one given by Ter-Krikorov [12].

If $v_0 = 0$, (5.13) reduces to the familiar result

$$f = q \ \mathrm{sech}^2 \frac{x_1}{2h} \sqrt{\frac{3q}{h}} \ . \qquad (5.15)$$

It is easy to compare this with the results corresponding to $v_c(y) = \zeta(y+1)$ where the dimensionless vorticity ζ is a positive constant. The critical speeds for constant vorticity are given by

$$\int_{-1}^{0} \frac{dy}{[c_0 + \zeta(y+1)]^2} = \frac{1}{gh}$$

that is

$$c_0(c_0 + \zeta) = gh$$

$$c_0 = \frac{-\zeta \pm \sqrt{\zeta^2 + 4gh}}{2}$$

or in terms of the mean speed $\overline{v}_o = \zeta/2$:

$$c_o = -\overline{v}_o \pm \sqrt{gh + \overline{v}_o^2} \quad .$$

Also

$$m_o = bgh \int_{-1}^{0} [c_o + \zeta(y+1)]^2 \left\{ \int_{-1}^{y} \frac{d\xi}{[c_o + \zeta(\xi+1)]^2} \right\}^2 dy$$

$$= \frac{bgh}{3c_o^2}$$

and we have

$$\frac{\kappa}{m_o} = 3ghc_o^2 \int_{-1}^{0} \frac{dy}{[c_o + \zeta(y+1)]^4}$$

$$= \frac{3ghc_o^2}{[c_o + \zeta(y_1+1)]^2} \int_{-1}^{0} \frac{dy}{[c_o + \zeta(y+1)]^2}$$

$$\frac{\kappa}{m_o} = \frac{3c_o^2}{[c_o + \zeta(y_1+1)]^2} \, , \qquad -1 < y < 0$$

by using a mean value theorem. If $c_o > 0$, then

$$\frac{\kappa}{m_o} < 3 \, .$$

This means that a wave of given amplitude advancing upstream on a stream of constant vorticity has less curvature, is broader, than its counterpart on a stream of zero vorticity. If $c_o < 0$, c_o must be less than $-\zeta$, and $|c_o| > \zeta$; so for this case

$$\frac{\kappa}{m_o} = \frac{3|c_o|^2}{[|c_o| - \zeta(y_1+1)]^2} > 3$$

which shows that the wave moving downstream is sharper than its counterpart.

The velocity c of the wave is given by

$$-c = c_o + \frac{\kappa q\sqrt{\frac{g}{d}}}{2\iint\limits_D \frac{dzdy}{u_o^3}} + o(q) .$$

Since

$$\kappa = b(gh)^2 \int_{-1}^{0} \frac{dy}{[c_o + \zeta(y+1)]^4}$$

$$= \frac{b(3gh + \zeta^2)}{3gh}$$

and

$$2\iint\limits_D \frac{dzdy}{u_o^3} = 2b(gh)^{3/2} \int_{-1}^{0} \frac{dy}{[c_o + \zeta(y+1)]^3}$$

$$= \frac{b(2c_o + \zeta)}{\sqrt{gh}}$$

we have

$$-c = c_o + \frac{q(\zeta^2 + 3gh)}{3h(\zeta + 2c_o)} .$$

If $c_o > 0$,

$$-c = \sqrt{gh + \overline{v}_o^2} - \overline{v}_o + \frac{q(gh + \frac{4\overline{v}_o^2}{3})}{2h\sqrt{gh + \overline{v}_o^2}} .$$

If $c_o < 0$,

$$-c = -\sqrt{gh+\bar{v}_o^2} - \bar{v}_o - \frac{q(gh+\frac{4\bar{v}_o^2}{3})}{2h\sqrt{gh+\bar{v}_o^2}} .$$

In closing we remark that although we have confined our attention to solitary waves, we can easily turn the analysis into one on cnoidal waves by relaxing the conditions (4.34) and (4.48). For cnoidal waves we find the equation

$$N_o\eta_1'''(x) = N_1\eta_1(x)\eta_{1x}(x) + N_2\eta_1'(x) ,$$

that is, equation (4.47) but not with the same coefficients. The general integral of this equation satisfies

$$N_o[\eta_1'(x)]^2 = \frac{N_1}{3}\eta_1^3 + N_2\eta_1^2 + k_1\eta_1 + k_2 .$$

This leads to periodic waves of the form

$$\eta_1 = A + B\ \mathrm{cn}(x,k)$$

where cn(x,k) is Jacobi's elliptic function. We note finally that our method of analysis is applicable to other situations involving long rotational and irrotational waves in straight channels. For instance, our method could be used to study solitary waves in a medium of two layers, with different densities, contained in a channel with arbitrary cross section.

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